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STABILITY OF LOTKA–VOLTERRA SYSTEM

A. A. Soliman UDC 517.929

Abstract. The stability of Lotka–Volterra system has been discussed by many authors for two and three species. In this paper, we will discussed the notion of stability for a Lotka–Volterra system with four species. Some criteria and results are given. Our technique depends on the Lyapunov–Razumikhin method.

CONTENTS

1. Introduction

Consider a Kolmogorove-type differential system

$$x_i' = x_i f_i(x_1, x_2, x_3, \dots, x_n), \quad i = 1, 2, 3, \dots, n. \tag{1.1}$$

The Lotka–Volterra system is obtained from (1.1) for

$$f_i(x_1, x_2, x_3, \dots, x_n) = \left(a_i - \sum_{j=1}^{n} b_{ij} x_j\right), \qquad x_1, x_2, x_3, \dots, x_n \in \mathbb{R}_n^+,$$

i.e.,

$$x_i' = x_i \left(a_i - \sum_{j=1}^{n} b_{ij} x_j\right), \qquad x_1, x_2, x_3, \dots, x_n \in \mathbb{R}_n^+, \quad i = 1, 2, 3, \dots, n. \tag{1.2}$$

The Lotka–Volterra system is of considerable interest (see [1, 3–9, 11, 14, 16]). Cushing [1], Mangel [10], Redheffer [14], and Tackeuchi [16] studied system (1.2) in the particular case of two species $i = 1, 2$. May et al. [8], Schuster [15], and Pimbley [12] discussed system (1.1) in the case of three species $i = 1, 2, 3$, where $f_i \in C[\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \dots \times \mathbb{R}^+, \mathbb{R}^n]$, $f_i(x_1, x_2, x_3, \dots, x_n) > 0$, $f_i(0, 0, 0, \dots, 0) \neq 0$, $x_i > 0$, and $x_i(0) = x_{0i} > 0$. Gopalsamy [3] used sufficient conditions for the asymptotic stability of the zero solution of the system

$$x_i' = \sum_{j=1}^{2} b_{ij} x_j(t - \tau_{ij}), \quad i = 1, 2,$$

and obtained a sufficient condition for the global asymptotic stability of the positive equilibrium of a competition system in the case of two species with finite delay

$$x_i' = x_i \left(a_i - \sum_{j=1}^{2} b_{ij} x_j(t - \tau_{ij})\right).$$

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Now, we will discuss the stability of system (1.1)

$$x_i' = x_i\left(a_i - \sum_{j=1}^{4} b_{ij}x_j\right) \tag{1.3}$$

in the case of four species $i = 1, 2, 3, 4$, and

$$x_i' = x_i\left(a_i - \sum_{j=1}^{4} b_{ij}x_j(t-\tau_{ij})\right), \tag{1.4}$$

where $x_i \geq 0$, with parameters $a_i > 0$ and $b_{ij} > 0$.

We define a Lyapunov function $V(t,x) \in C[J \times \mathbb{R}^n, \mathbb{R}]$ and the function

$$D^+V(t) = \lim_{\delta \to 0^+} \sup \frac{1}{\delta}\big[V(t+\delta, x+\delta f(t,x)) - V(t,x)\big].$$

The following definitions will be needed.

Definition 1.1 (see [13]). A function $\phi(r)$ belongs to the class $\mathcal{K}$ if $\phi(r) \in C[(0,\rho), \mathbb{R}^+]$, $\phi(0) = 0$, and $\phi(r)$ is strictly monotonically increasing in r.

Definition 1.2 (see [13]). The zero solution of system (1.1) is said to be stable if for $\varepsilon > 0$, there exist positive numbers $\delta_1(\varepsilon) > 0$ such that for any solution $x(t, t_0, x_0)$ of system (1.1), the inequality

$$\|x(t, t_0, x_0)\| < \varepsilon$$

holds, provided that

$$\|x_0\| \leq \delta_1 \Longrightarrow \|x\| < \varepsilon, \quad t \leq t_0.$$

Definition 1.3 (see [13]). The zero solution of system (1.1) is said to be uniformly stable in variation if there exist positive numbers $\delta_1 > 0$ and $M > 0$ such that

$$\|\Phi(t, t_0, x_0)\| \leq M,$$

whenever

$$\|x_0\| \leq \delta_1 \quad \text{for} \quad t \geq t_0 \geq 0,$$

where $\Phi(t, t_0, x_0)$ is the fundamental matrix solution of the system

$$y' = f_x(t, x(t, t_0, x_0))y \tag{1.5}$$

given by

$$\Phi(t, t_0, x_0) = \frac{\partial x(t, t_0, x_0)}{\partial x_0}, \qquad \Phi(t_0, t_0, x_0) = I,$$

where $x(t, t_0, x_0)$ is a solution of (1.1), $f_x(t,x) = \dfrac{\partial f(t,x)}{\partial x}$, and I is the indentity matrix.

Definition 1.4 (see [13]). The zero solution of system (1.1) is said to be asymptotically stable if it is stable and for $\varepsilon > 0$, there exist positive numbers $\delta_1 > 0$ and $T(\varepsilon) > 0$ such that for any solution $x(t, t_0, x_0)$ of system (1.1), the inequality

$$\|x(t, t_0, x_0)\| < \varepsilon$$

holds, provided that

$$\|x_0\| \leq \delta_1, \quad t \geq t_0 + T.$$

2. Immune System and Lyapunov Functional

In this section, we study the stability of the following system:

$$x_i' = x_i\left(a_i - \sum_{j=1}^{4} b_{ij}x_j\right). \tag{2.1}$$

The linearized system of (1.2) is given by

$$z' = Az, \tag{2.2}$$

where $z = (x_1, x_2, x_3, x_4)^T$ and A is the (4×4)-variational matrix from (2.1) at a critical point $(x_1^*, x_2^*, x_3^*, x_4^*)$ given by

$$A = \begin{pmatrix} x_1^*\dfrac{\partial f_1}{\partial x_1} + f_1 & x_1^*\dfrac{\partial f_1}{\partial x_2} & x_1^*\dfrac{\partial f_1}{\partial x_3} & x_1^*\dfrac{\partial f_1}{\partial x_4} \\ x_2^*\dfrac{\partial f_2}{\partial x_1} & x_2^*\dfrac{\partial f_2}{\partial x_2} + f_2 & x_2^*\dfrac{\partial f_2}{\partial x_3} & x_2^*\dfrac{\partial f_2}{\partial x_4} \\ x_3^*\dfrac{\partial f_3}{\partial x_1} & x_3^*\dfrac{\partial f_3}{\partial x_2} & x_3^*\dfrac{\partial f_3}{\partial x_3} + f_3 & x_3^*\dfrac{\partial f_3}{\partial x_4} \\ x_4^*\dfrac{\partial f_4}{\partial x_1} & x_4^*\dfrac{\partial f_4}{\partial x_2} & x_4^*\dfrac{\partial f_4}{\partial x_3} & x_4^*\dfrac{\partial f_4}{\partial x_4} + f_4 \end{pmatrix}.$$

Thus,

$$A = \begin{pmatrix} x_1^*b_{11} + f_1 & x_1^*b_{12} & x_1^*b_{13} & x_1^*b_{12}b_{14} \\ x_2^*b_{21} & x_2^*b_{22} + f_2 & x_2^*b_{23} & x_2^*b_{24} \\ x_3^*b_{31} & x_3^*b_{32} & x_3^*b_{33} + f_3 & x_3^*b_{34} \\ x_4^*b_{41} & x_4^*b_{42} & x_4^*b_{43} & x_4^*b_{44} + f_4 \end{pmatrix},$$

where

$$f_1 = a_1 - (b_{11}x_1 + b_{12}x_2 + b_{13}x_3 + b_{14}x_4), \qquad f_2 = a_2 - (b_{21}x_1 + b_{22}x_2 + b_{23}x_3 + b_{24}x_4)),$$
$$f_3 = a_3 - (b_{31}x_1 + b_{32}x_2 + b_{33}x_3 + b_{34}x_4)), \qquad f_4 = a_4 - (b_{41}x_1 + b_{42}x_2 + b_{43}x_3 + b_{44}x_4)).$$

The eigenvalues λ_i of A are obtained by solving the characteristic equation $\|A - \lambda I\| = 0$, i.e.,

$$\lambda^4 - p_1\lambda^3 + p_2\lambda^2 - p_3\lambda + Q = 0,$$

where p_i are constants, $i = 1, 2, 3$, and Q is the determinant of A. Since $f_i(0,0,0,0,) \neq 0$, $f_i(x_1, x_2, x_3, x_4) > 0$, $x_i \geq 0$.

By a simple computation, we obtain that the critical points are $e_0(0,0,0,0)$ and $e_1(x_1, x_2, x_3, x_4)$, where $x_i = \triangle_i/\triangle$, $i = 1, 2, 3, 4$, where $\triangle$ and $\triangle_i$, $i = 1, 2, 3, 4$, are defined in the appendix. For the critical point $e_0 = (0,0,0,0)$, the variational matrix A becomes

$$A_0 = \begin{pmatrix} f_1(e_0) & 0 & 0 & 0 \\ 0 & f_2(e_0) & 0 & 0 \\ 0 & 0 & f_3(e_0) & 0 \\ 0 & 0 & 0 & f_4(e_0) \end{pmatrix} = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix}$$

and has eigenvalues $\lambda_i = a_i > 0$, $i = 1, 2, 3, 4$. From [13, Theorem 3.2.3], we obtain that $e_0(0, 0, 0, 0)$ is an unstable node.

For $e_1(x_1, x_2, x_3, x_4)$, by a simple computation, we obtain the following form of A_1:

$$A_1 = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 \end{pmatrix},$$

where $\alpha_1 = x_1^* b_{11} + f_1$, $\alpha_2 = x_1^* b_{12}$, $\alpha_3 = x_1^* b_{13}$, $\alpha_4 = x_1^* b_{12} b_{14}$, $\beta_1 = x_2^* b_{21}$, $\beta_2 = x_2^* b_{22} + f_2$, $\beta_3 = x_2^* b_{23}$, $\beta_4 = x_2^* b_{24}$, $\gamma_1 = x_3^* b_{31}$, $\gamma_2 = x_3^* b_{32}$, $\gamma_3 = x_3^* b_{33} + f_3$, $\gamma_4 = x_3^* b_{34}$, $\eta_1 = x_4^* b_{41}$, $\eta_2 = x_4^* b_{42}$, $\eta_3 = x_4^* b_{43}$, and $\eta_4 x_4^* b_{44} + f_4$.

Remark 2.1. Schuster [15] studied system (2.1) in the particular case $x_4 = 0$, $a_i = b_{ii} = 1$, $b_{12} = b_{23} = b_{34} = a$, $b_{13} = b_{24} = b_{31} = b$, and $b_{21} = b_{32} = c$. Also, Pimbley [12] discussed system (2.1) in the particular case $x_4 = 0$, $a_1 = 1 - \lambda_1$, $a_2 = -\lambda_2$, $a_3 = -\lambda_3$, $b_{11} = k\lambda_1$, $b_{12} = k(\alpha_1 - \lambda_1)$, $b_{13} = kn\lambda_1$, $b_{21} = -k(\alpha_2 - \lambda_2)$, $b_{22} = -k\lambda_2$, $b_{23} = -kn\lambda_2$, $b_{31} = k(\alpha_3 - \lambda_3)$, $b_{32} = -k\lambda_3$, and $b_{33} = -kn\alpha_3\lambda_3$.

Example 2.1. Consider a model four-component immune system

$$\begin{cases} u_1' = u_1 f_1(u_1, u_2, u_3, u_4), \\ u_2' = u_2 f_2(u_1, u_2, u_3, u_4), \\ u_3' = u_3 f_3(u_1, u_2, u_3, u_4), \\ u_4' = u_4 f_4(u_1, u_2, u_3, u_4), \end{cases} \tag{2.3}$$

where

$$\begin{cases} f_1 = \lambda_1 + k\lambda_1 u_1 - k(\eta_1 - \lambda_1)u_2 + kn\lambda_1 u_3 + kn\lambda_1 u_4, \\ f_2 = -\lambda_1 - k(\eta_2 + \lambda_2)u_1 - k\lambda_2 u_2 - kn\lambda_2 u_3 + kn\lambda_2 u_4 + k\gamma u_3, \\ f_3 = -\lambda_3 + k(\eta_3 - \lambda_3)u_1 - k\lambda_3 u_2 - kn\lambda_3 u_3 + \lambda_3 u_4 - k\eta_3 u_1 \cdot u_3, \\ f_4 = -\lambda_4 + k(\eta_4 + \lambda_4)u_1 - k\lambda_4 u_2 - kn\lambda_4 u_3 - k\lambda_4 u_4 \\ \qquad\qquad + s[1 + k(u_1 + u_2 + u_3 + nu_4)] - k\dfrac{\eta_4}{\theta} u_2 \cdot u_4, \end{cases} \tag{2.4}$$

where u_1, u_2, u_3, u_4 denote the concentrations of antigen, antibody, antibody production, and antigen production cells, respectively, η_1 denotes the elimination rate of antigen, λ_1 is the increasing rate of antigen when antibody is absent, η_2 denotes the rate production of antibody which stimulates limiting of antigen to antibody, λ_2 denotes the decreasing rate of antibody when antigen is absent, and λ_3 denotes the rate of elimination of antibody production cells in absence of antigens, λ_4 denotes the inreasing rate of antigen production cells under decreasing antibody production, θ is the concentration level that can be exceded, s is a constant source term, n is the number of receptors per cell, and θ must be sufficiently large.

We note that if s and $k\gamma$ are sufficiently small, system (2.4) becomes

$$\begin{cases} u_1' = u_1(\lambda_1 + k\lambda_1 u_1 - k(\eta_1 - \lambda_1)u_2 + kn\lambda_1 u_3 + kn\lambda_1 u_4), \\ u_2' = u_2(-\lambda_1 - k(\eta_2 + \lambda_2)u_1 - k\lambda_2 u_2 - kn\lambda_2 u_3 + kn\lambda_2 u_4), \\ u_3' = u_3(-\lambda_3 + k(\eta_3 - \lambda_3)u_1 - k\lambda_3 u_2 - kn\lambda_3 u_3 + \lambda_3 u_4 - k\eta_3 u_1 \cdot u_3), \\ u_4' = u_4(-\lambda_4 + k(\eta_4 + \lambda_4)u_1 - k\lambda_4 u_2 - kn\lambda_4 u_3 - k\lambda_4 u_4 - k\dfrac{\eta_4}{\theta} u_2 \cdot u_4). \end{cases} \tag{2.5}$$

It is clear that the five critical points on the boundary $\mathbb{R}^4$ are

$$(0,0,0,0),\quad \left(\frac{-1}{k},0,0,0\right),\quad \left(0,\frac{-1}{k},0,0\right),\quad \left(0,0,\frac{-1}{kn},0\right),\quad \left(0,0,0,-\frac{1}{kn}\right),$$

and $(A,0,0,0)$, $(0,0,B,0)$, $(0,C,0,0)$, where

$$A=\frac{-\lambda_2}{k(\eta_2+\lambda_2)},\qquad B=\frac{-\lambda_3}{k(\eta_3-\lambda_3)},\qquad C=\frac{\lambda_1}{k(\eta_1-\lambda_1)}.$$

Now, we assume that θ is sufficiently large (in this case, the antibody production is limited). For the surface $u_4=0$, it is clear that $(0,0,0)$, $(A,0,0)$, and $(0,0,B)$ are sources and, therefore, they all are unstable (see Fig. 1), while

$$(0,C,0),\quad \left(-\frac{1}{k},0,0\right),\quad \text{and}\quad \left(0,0,-\frac{1}{kn}\right)$$

are sinks and, therefore, they all are stable (see Fig. 2).

Theorem 2.1. *The zero solution of* (2.1) *is uniformly asymptotically stable if* $a_i>0$ *and* $b_{ij}>0$, $i,j=1,2,3,4$.

Proof. Let $b(\|x\|)\geq\|x\|$, $b\in\mathcal{K}$, and $V(x)=\|x\|=\|(x_1,x_2,x_3,x_4)\|$ be a Lyapunov function. It is clear that $V(x)$ is a positive definite and decreasing function for $x_i\geq 0$, $i=1,2,3,4$. Then

$$V(x)=\|x\|\leq b(\|x\|),\quad b\in\mathcal{K}.$$

Thus,

$$\begin{aligned}V'(x)&=\|x'\|=\|(x_1'+x_2'+x_3'+x_4')\|\\&=a_1x_1+a_2x_2+a_3x_3+a_4x_4-(b_{11}x_1^2+b_{22}x_2^2+b_{33}x_3^2+b_{44}x_4^2)\\&-(b_{32}+b_{21}+b_{23}+b_{12}+b_{34})[x_1x_2+x_2x_3+x_3x_4+x_4x_1]-2(b_{31}+b_{24}+b_{13})[x_1x_3+x_2x_4],\end{aligned}$$

and we obtain

$$V'=V-V^2-(b_{11}x_1^2+b_{22}x_2^2+b_{33}x_3^2+b_{44}x_4^2)-2(b_{31}+b_{24}+b_{13})[x_1x_3+x_2x_4]<0.$$

Following [13, Theorem 5.2.2], the zero solution of system (2.1) is uniformly asymptotically stable, and the proof is complete.

We consider the following system (2.1):

$$x_i'=x_i\left(a_i-\sum_{j=1}^{4}b_{ij}x_j(t-\tau_{ij})\right),\quad \tau_{ij}\in[0,\infty).$$

□

Theorem 2.2. *Let the integrals*

$$\int_{t=\tau_{ij}}^{t}a_ix_i(s+\tau_{ij})\,ds\quad\text{and}\quad\int_{t=\tau_{ij}}^{t}b_{ij}x_i(s+\tau_{ij})x_j(s+\tau_{ij})\,ds \tag{2.6}$$

exist and converge on $[0,\infty)$, *and let* $\mu_i=2(a_i+1)>1$ *be a scalar such that*

$$\mu_ix_i<\sum_{i=1}^{n}Q_i+2\sum_{i=1}^{n}\int_{t=\tau_{ij}}^{t}\big[a_ix_i(s+\tau_{ij})+b_{ij}x_i(s+\tau_{ij})x_j(s+\tau_{ij})\big]ds<0. \tag{2.7}$$

Then the zero solution of (2.1) *is uniformly asymptotically stable.*

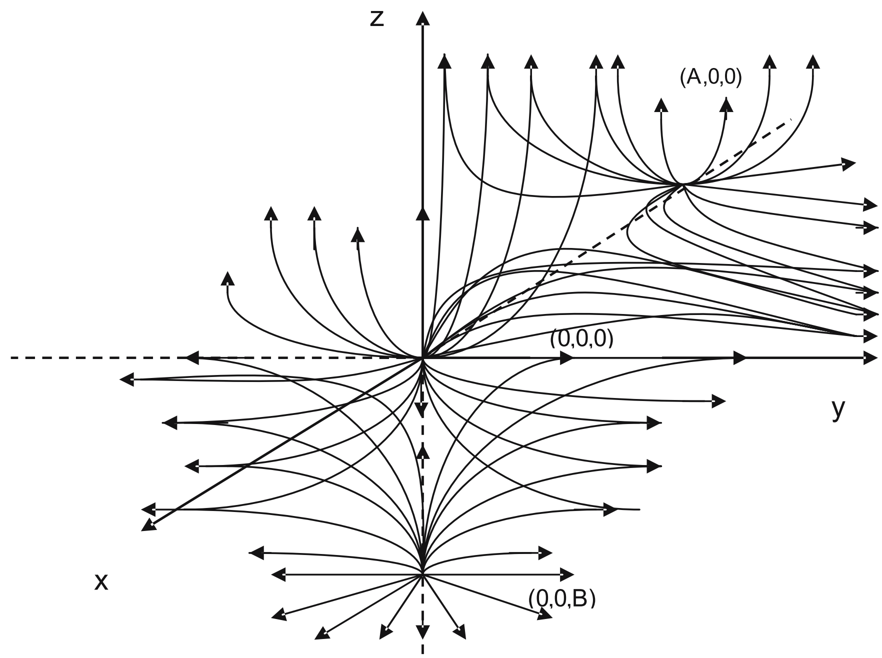

Fig. 1 Unstable points

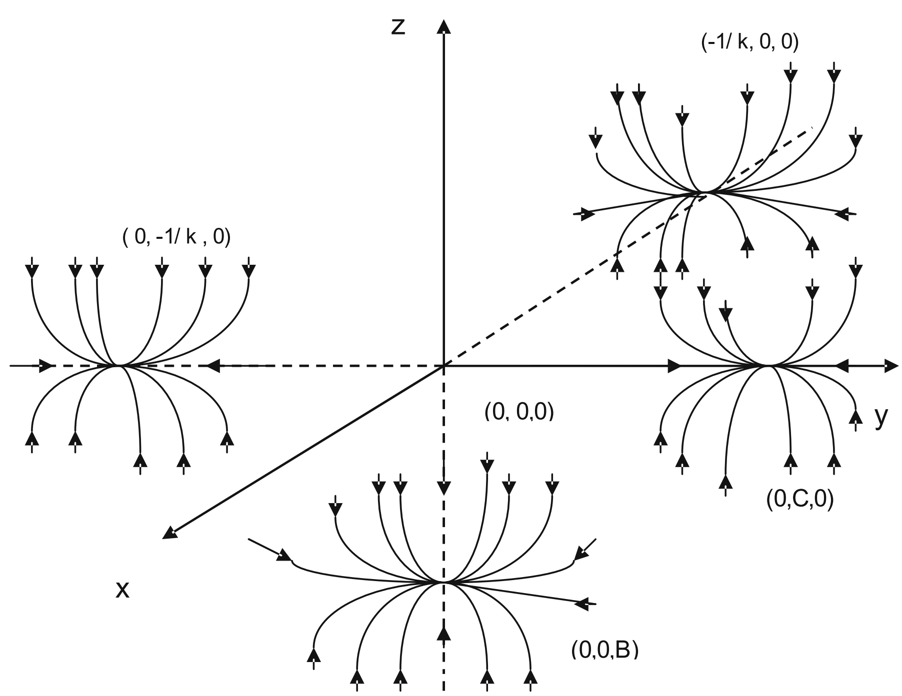

Fig. 2 Stable points

Proof. Define a Lyapunov functional

$$V = \left[\sum_{i,j=1}^{4} \left(x_i - \int_{t=\tau_{ij}}^{t} a_i x_i(s+\tau_{ij})\, ds \right) - \sum_{i,j=1}^{4} \int_{t=\tau_{ij}}^{t} b_{ij} x_i(s+\tau_{ij}) x_j(s+\tau_{ij})\, ds \right]^2 .$$

Then V' is

$$V' = 2\sqrt{V}.\frac{d}{dt}\left[\sum_{i,j=1}^{4}\left(x_i - \int\limits_{t=\tau_{ij}}^{t} a_i x_i(s+\tau_{ij})ds\right) - \sum_{i,j=1}^{4}\int\limits_{t=\tau_{ij}}^{t} b_{ij}x_i(s+\tau_{ij})x_j(s+\tau_{ij})ds\right].$$

For $a_i > 0, b_{ij} > 0$, and $\tau_{ij} \in [0,\infty)$, $i = 1,2,3,4$, we obtain

$$V'(x_1,x_2,x_3,x_4) \leq 0.$$

Since $V(0,0,0,0) = 0$, following [13, Theorem 5.2.2], we obtain that the zero solution of (2.1) is uniformly asymptotically stable, and the proof is complete. ☐

3. Lyapunov–Razumikhin and Krasovskii's Methods

In this section, we study the stability of the zero solution of system (1.3) with finite delay via Lyapunov–Razumikhin's technique [13], and Krasovskii's method to construct Lyapunov function [13].

Let (x^*, y^*, z^*, w^*) be a critical point of system (1.3) satisfying

$$\begin{cases} x_i^*\left(a_i - \sum\limits_{j=1}^{4} b_{ij}x_i^*\right) = 0, \qquad y_i^*\left(a_i - \sum\limits_{j=1}^{4} b_{ij}y_i^*\right) = 0, \\ z_i^*\left(a_i - \sum\limits_{j=1}^{4} b_{ij}z_i^*\right) = 0, \qquad w_i^*\left(a_i - \sum\limits_{j=1}^{4} b_{ij}w_i^*\right) = 0. \end{cases} \tag{3.1}$$

Now, the linearized system corresponding to system (1.3) is obtained by using the transformations $x_1 = x^* + \varepsilon u_1$, $x_2 = y^* + \varepsilon u_2$, $x_3 = z^* + \varepsilon u_3$, $x_4 = w^* + \varepsilon u_4$. Then system (1.3) becomes

$$\begin{cases} u_1' = \alpha u_1 - x^*b_{12}u_2 - x^*b_{13}u_3 - x^*b_{14}u_4, \\ u_2' = -y^*b_{21}u_1 + \beta u_2 - y^*b_{23}u_3 - y^*b_{24}u_4, \\ u_3' = -z^*b_{31}u_1 - z^*b_{32}u_2 + \gamma u_3 - y^*b_{34}u_4, \\ u_4' = -w^*b_{41}u_1 - w^*b_{42}u_2 - w^*b_{43}u_3 + \theta u_4, \end{cases} \tag{3.2}$$

where

$$\begin{aligned} \alpha &= a_1 - 2x^*b_{11} - y^*b_{12} - z^*b_{13} - w^*b_{14}, \\ \beta &= a_2 - x^*b_{21} - 2y^*b_{22} - z^*b_{23} - w^*b_{24}, \\ \gamma &= a_3 - x^*b_{31}u_2 - y^*b_{32} - 2z^*b_{33} - w^*b_{34}, \\ \theta &= a_4 - x^*b_{41} - y^*b_{42} - z^*b_{43} - 2w^*b_{44}. \end{aligned}$$

We can write (3.2) as

$$Z' = EZ, \tag{3.3}$$

where

$$E = \begin{pmatrix} b_{11} & -x^*b_{12} & -x^*b_{13} & -x^*b_{14} \\ -y^*b_{21} & b_{22} & -y^*b_{23} & -y^*b_{24} \\ -z^*b_{31} & -z^*b_{32} & b_{33} & -z^*b_{34} \\ -w^*b_{41} & -w^*b_{42} & -w^*b_{43} & b_{44} \end{pmatrix}, \qquad Z = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix},$$

where $b_{11} = \alpha$, $b_{22} = \beta$, $b_{33} = \gamma$, $b_{44} = \theta$.

Following [13], by using Krasovskii's method, we define

$$V = Z^T BZ. \tag{3.4}$$

Thus,

$$V' = Z^T(E^TB + BE)Z.$$

Since $E_0 = E(0,0,0,0) = \mathrm{diag}\{a_i\}$, $a_i > 0$, $i = 1,2,3,4$, we have that E_0 is unstable and, therefore, it does not satisfy the matrix equation

$$E^TB + BE = -I, \tag{3.5}$$

where I is the identity matrix. Since V and V' are positive-definite functions, the zero solution of (3.2) is an unstable node.

Example 3.1. We consider system (3.2) with

$$E = \begin{pmatrix} 3 & 0 & 1 & 1 \\ -2 & 3 & -1 & 0 \\ -4 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 1 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We find that the eigenvalues of E are $1,3,4,4$. Therefore, B is unstable, since all eigenvalues are positive and B does not satisfy (3.5), where

$$B = \begin{pmatrix} 1 & -2 & 2 & 2 \\ -2 & -1 & -1 & 0 \\ 5 & -1 & 1 & 0 \\ 1 & -2 & 5 & 1 \end{pmatrix}.$$

Since $V = Z^TBZ$, we have $V' > 0$. Therefore, the zero solution of system (3.3) is unstable.

Example 3.2. We consider the Lorentz system

$$\begin{cases} u_1' = \sigma(u_2 - u_1), \\ u_2' = \rho u_1 - u_2 - u_1u_2, \\ u_3' = -\beta u_3 - u_1u_2, \end{cases} \tag{3.6}$$

where σ, ρ, and β are real positive parameters denoting the Prandtl number, the Rayleigh number, and a geometric factor, respectively. The state variables u_1, u_2, and u_3 represent the velocity of the fluid layer and spatial temperature distribution in the fluid layer under gravity.

Lorentz system has the following properties: if $\rho \in (\beta, \rho)$, there exists one unstable point – the origin point and there are two stable points

$$\Big(\sqrt{\beta(\rho-1)}, \sqrt{\beta(\rho-1)}, (\rho-1)\Big), \qquad \Big(-\sqrt{\beta(\rho-1)}, -\sqrt{\beta(\rho-1)}, (\rho-1)\Big).$$

If ρ is larger than the number $\rho_1 \in (\beta, \rho)$, then there are no stable points and trajectories of the system have a chaotic behavior.

System (3.6) was studied by other authors from their different points of view (see [4–7, 14–16]).

We study the zero solution of Lorentz system (3.6). Choose the Lyapunov function

$$V(t, U) = \frac{1}{2}(u_1^2 + u_2^2 + u_3^2),$$

which is positive definite and decreasing, where $U = (u_1, u_2, u_3)^T$. Along solutions of (3.6), we have

$$V'(t, U) = -\sigma u_1^2 - u_2^2 + (\sigma + \rho)u_1u_2 - \beta u_3^2 \le \mu V(t, U), \tag{3.7}$$

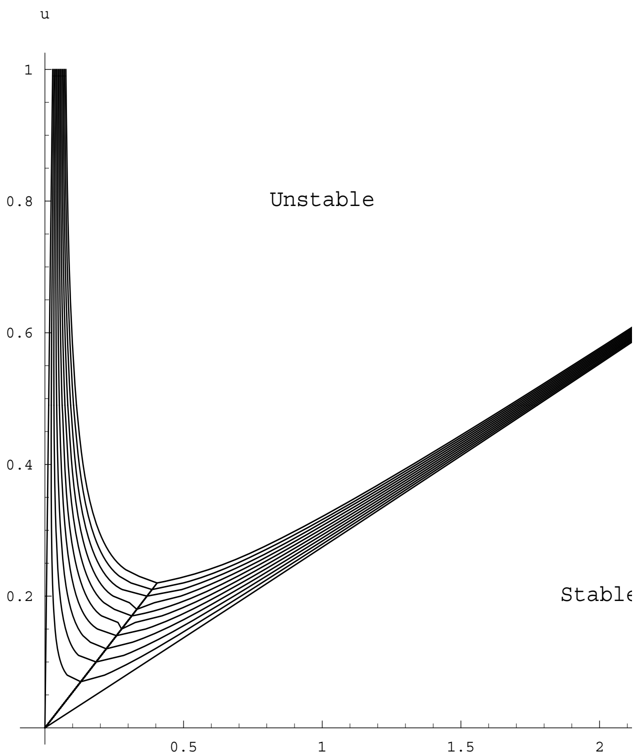

Fig. 3

where

$$\mu = \max\left\{ -(\sigma+1) - \sqrt{(\sigma-1)^2 + (\sigma+\rho)^2}, -2\beta \right\}. \tag{3.8}$$

Hence $V'(t, U) \leq 0$ and, therefore, the zero solution of (3.6) is asymptotically stable. Therefore all trajectories converge to the origin (see Fig. 3).

Consider the system

$$x' = Ax(t) + \int_0^t k(t-s)x(s)\, ds, \tag{3.9}$$

where $A = [a_{ij}]$ and $K = [k_{ij(t)}]$ are real and functional $(n \times n)$-matrices, K belongs to $L^1[0, \infty)$.

Now, we define the following class.

Definition 3.1. A function $b(r)$ is said to be of the class $\mathcal{H}$ if $b \in \mathbb{R}^n$ is such that

$$\|b(s)\| \leq N\|b(r)\|, \quad N > 1, \quad -\infty < s < r < \infty.$$

Theorem 3.1. *Let the matrix $A = [a_{ij}]$ in (3.6) be a real $(n \times n)$-matrix and let*

$$\int_0^t k(t-s)ds = k_0, \tag{3.10}$$

where $k = [k_{ij}(t)]$ is an $(n \times n)$-matrix function which belongs to $L^1[0, \infty)$ and $k_0 > 0$. Then the zero solution of (3.6) is asymptotically stable.

Proof. Following [13], we define a Lyapunov function $V(x)$ as follows:

$$V(x) = x^T Bx, \tag{3.11}$$

where B is a symmetric constant $(n \times n)$-matrix such that

$$A^T B + BA = -I, \tag{3.12}$$

where I is the identity matrix, and Eq. (3.12) has a positive-definite matrix solution B. The time derivative of $V(x)$ along the solution of (3.5) together with (3.11) yield

$$V'(x) = (x^T)'Bx + x^T Bx'$$

$$= \left[x^T A^T + \int_0^t k(t-s)x^T(t)\,ds\right]Bx + x^T B\left[Ax(t) + \int_0^t k(t-s)x(s)\,ds\right]$$

$$= x^T A^T Bx + x^T BAx + \left[\int_0^t k(t-s)x(s)\,ds\right]Bx + x^T B\left[\int_0^t k(t-s)x(s)\,ds\right]$$

$$= x^T(A^T B + BA)x + \left[\int_0^t k(t-s)x(s)\,ds\right]Bx + x^T B\left[\int_0^t k(t-s)x(s)\,ds\right]$$

$$\leq -x^T x + \left[\int_0^t k(t-s)x(s)\,ds\right]Bx + x^T B\left[\int_0^t k(t-s)x(s)ds\right]$$

$$\leq -\|x\|^2 + 2N\|B\|\|x\|^2 \int_0^t k\|k(t-s)x(s)\|ds$$

$$\leq -\|x\|^2 + 2N\|B\|\|x\|^2 \int_0^t k(t-s)\,ds = -\big[1 - 2Nk_0\|B\|\big]\,\|x\|^2,$$

and, therefore, $V'(x) \leq 0$ for $k_0 = 1/(2N\|B\|)$; therefore, by [13, Theorem 5.2.2], the zero solution of (3.5) is asymptotically stable. □

4. Appendix

System (2.2) is equivalent to the following system:

$$\frac{d}{dt}\left[\sum_{i,j=1}^{4}\left(x_i - \int_{t=\tau_{ij}}^{t} a_i x_i(s+\tau_{ij})\,ds\right) - \sum_{i,j=1}^{4}\int_{t=\tau_{ij}}^{t} b_{ij}x_i(s+\tau_{ij})x_j(s+\tau_{ij})\,ds\right]$$

$$= \sum_{i,j=1}^{4}[x_i(1-a_i x_i) - a_i x_i(t+\tau_{ij})] - \sum_{i,j=1}^{4}[b_{ij}x_i(t+\tau_{ij}x_j(t) - b_{ij}x_i(t+\tau_{ij})x_j(t+\tau_{ij})];$$

$$\begin{aligned}\triangle = {}& 2b_{11}b_{13}b_{23}b_{44} + b_{14}^2 b_{23}^2 + 2b_{12}b_{14}b_{23}b_{24} + 2b_{11}b_{23}b_{24}b_{34} + b_{13}^2(b_{34})^2 \\ & + 2b_{13}b_{14}b_{22}b_{34} + b_{13}^2(b_{24})^2 - 2b_{12}b_{14}b_{23}b_{34} - b_{11}(b_{24})^2 b_{33} - 2b_{13}b_{14}b_{23}b_{24} \\ & - (b_{14})^2 b_{22}b_{33} - b_{12}b_{13}b_{24}b_{34} - b_{13}^2 b_{22}b_{44} - b_{11}b_{23}^2 b_{44} - b_{12}^2 b_{33}b_{44} \\ & - b_{11}b_{22}b_{33}b_{44} - b_{11}b_{23}^2 b_{22},\end{aligned}$$

$$\begin{aligned}\triangle_1 = {}& [-a_4 b_{13}b_{23}b_{24} - a_4 b_{14}b_{22}b_{33} - a_3 b_{14}b_{23}b_{24} - a_1 b_{14}b_{24}^2 b_{33} - a_1 b_{22}b_{33}b_{44} \\ & - a_3 b_{13}b_{22}b_{44} + a_4 b_{12}b_{24}b_{33} + a_4 b_{13}b_{22}b_{34} + a_3 b_{14}b_{22}b_{34} + a_4 b_{14}b_{23}^2 \\ & + a_2 b_{14}b_{24}b_{33} + a_3 b_{13}b_{24}^2 + 2a_1 b_{23}b_{24}b_{34} + a_2 b_{12}b_{34}^2 + a_3 b_{12}b_{23}b_{44}\end{aligned}$$

$$+ a_2 b_{13} b_{23} b_{44} + a_3 b_{13} b_{24} b_{34}],$$

$$\triangle_2 = [-a_4 b_{13} b_{23} b_{14} + a_3 b_{14}^2 b_{23} + a_4 b_{13}^2 b_{24} - a_3 b_{14} b_{24}^2 b_{13} - a_4 b_{12} b_{33} b_{14} - a_2 b_{13} b_{14}^2 - a_4 b_{12} b_{13} b_{34} - a_3 b_{12} b_{14} b_{34} + 2a_4 b_{13} b_{14} b_{34} + a_4 b_{11} b_{23} b_{34} - a_1 b_{14} b_{23} b_{34} + a_3 b_{11} b_{24} b_{44} - a_2 b_{13}^2 b_{44} - a_3 b_{11} b_{23} b_{44} + a_1 b_{13} b_{23} b_{44} - a_2 b_{11} b_{33} b_{44},$$

$$\triangle_3 = [-a_4 b_{13} b_{22} b_{14} + a_3 b_{14}^2 b_{22} + a_4 b_{14} b_{23} b_{14} - a_2 b_{23} b_{14}^2 + a_4 b_{12} b_{13} b_{24} - 2a_3 b_{12} b_{14} b_{24} + a_2 b_{13} b_{24} b_{14} - a_4 b_{11} b_{23} b_{24} + a_1 b_{14} b_{23} b_{24} + a_3 b_{11} b_{24}^2 - a_1 b_{13} b_{24}^2 + a_4 b_{34} b_{12}^2 + a_2 b_{12} b_{14} b_{34} + a_4 b_{11} b_{22} b_{34} - a_1 b_{14} b_{22} b_{34} - a_2 b_{11} b_{24} b_{34} + a_1 b_{12} b_{24} b_{34}] + a_3 b_{12}^2 b_{44} - a_2 b_{11} b_{13} b_{44} - a_3 b_{11} b_{22} b_{44} - a_1 b_{13} b_{22} b_{44} + a_1 b_{12} b_{23} b_{44}],$$

$$\triangle_4 = [a_4 b_{13}^2 b_{22} - a_3 b_{14} b_{22} b_{13} - 2a_4 b_{12} b_{13} b_{23} - a_2 b_{13}^2 b_{24} - a_3 b_{11} b_{23} b_{24} + a_1 b_{13} b_{23} b_{24} + a_4 b_{33} b_{12}^2 + a_2 b_{12} b_{14} b_{33} + a_4 b_{11} b_{22} b_{33} - a_4 b_{11} b_{23}^2 + a_2 b_{14} b_{24} b_{33} + a_4 b_{11} b_{22} b_{33} + 2a_1 b_{23} b_{24} b_{34} + a_2 b_{12} b_{34}^2 + a_1 b_{14} b_{22} b_{33} + a_2 b_{11} b_{24} b_{33} - a_1 b_{12} b_{24} b_{33} - a_3 b_{12}^2 b_{34} - a_2 b_{12} b_{13} b_{14} + a_3 b_{11} b_{22} b_{34} - a_1 b_{11} b_{22} b_{34} + a_2 b_{11} b_{23} b_{34} + a_1 b_{12} b_{23} b_{34}].$$

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A. A. Soliman
E-mail: a_a_soliman hotmail.com
Department of Mathematics, Al-Jouf Teachers College,
Sakaka 269, Saudi Arabia